American Journal of Business Practice Vol.2, No.1 (Jan, 2025),

E-ISSN: 2997-934X

American Journal of Business Practice

https://semantjournals.org/index.php/AJBP

| Research Article

Unique Solvability of a Mixed Problem for a Fourth Order Time-Fractional Space Degenerate Partial Differential Equation

Mamanazarov Azizbek Otajon ugli

Associate Professor of Fergana State University, Fergana State University, Murabbiylar st. 19, Fergana city, 150100, Uzbekistan

Mirzagiyosova Sevinchbonu Mirzokhid qizi

Master Student of Fergana State University, Fergana State University, Murabbiylar st. 19, Fergana city, 150100, Uzbekistan

Abstract: In this work, in a rectangular domain, we study a mixed problem for a fourth order differential equation degenerating on the bound of the domain. By applying the method of separation of variables to the considered problem, a spectral problem for an ordinary differential equation has been obtained. Then, the Green's function of the spectral problem has been constructed, with the help of which it is equivalently reduced to the second kind Fregholm integral equation with a symmetric kernel. Using the theory of integral equations with symmetric kernels the existence and some properties of the eigenfunctions and eigenvalues of this spectral problem has been studied. The solution of the original problem has been written as the sum of a Fourier series with respect to the system of eigenfunctions of the spectral problem. Uniformly convergence of this series has been proved.

Key words: degenerate equation, a mixed problem, spectral method, Green's function, integral equation with symmetric kernel.

I. INTRODUCTION

The theory of differential equations is known for its long and rich history. Up until the last quarter of the 20th century, this field focused primarily on differential equations of integer order. However, with the advent and development of fractional calculus (encompassing both differential and integral analysis) in the late 20th century, researchers began exploring differential equations that incorporate fractional derivatives. Today, a significant number of scientific articles address initial, boundary, and spectral problems for differential equations – both ordinary and partial –

that involve fractional derivatives in various forms (see, for example, [1]-[4] and [5], and the references therein). Additionally, the books [6] and [7] have played a pivotal role in advancing this area of research.

We provide a brief review of studies related to this article's focus: a fourth-order partial differential equation featuring a fractional derivative of an unknown function with respect to the time variable.

The papers [8]-[10] investigate initial-boundary value problems for one-dimensional and two-dimensional fourth-order equations involving the Caputo fractional differentiation operator with respect to the time variable. Additionally, [10] addresses the inverse problem as well. Initial-boundary value problems for fourth-order equations incorporating fractional differentiation operators such as Hilfer, Dzhribashyan-Nersesyan, and Riemann-Liouville are examined in [11], [12], and [13], respectively. The direct and inverse problems for a mixed-type fourth-order equation with the Hilfer operator are studied in [14] and [15], respectively. Furthermore, we highlight the works [17] and [18], which deal with inverse problems concerning the determination of the order of a fractional derivative in the sense of Riemann-Liouville and Caputo. These are applied to subdiffusion equations and wave equations with an arbitrary positive operator possessing a discrete spectrum.

The papers mentioned above focus solely on non-degenerate equations. However, both local and nonlocal boundary value problems for degenerate partial differential equations containing fractional derivatives of an unknown function remain poorly unexplored. Investigating boundary value problems for such equations is highly significant, not only from a theoretical standpoint but also from a practical perspective. These equations, along with their associated problems, frequently arise in mathematical modeling across various fields, including gas and hydrodynamics, the theory of small surface bending, mathematical biology, and other scientific disciplines.

Initial-boundary value problems for degenerate equations with fourth-order fractional derivatives, involving both first and second time derivatives, were previously investigated in [18]-[20].

In the present paper we will prove the uniquely solvability of a mixed problem for a fourth order partial differential equation that degenerates on the boundary of the considered domain.

II. FORMULATION OF THE PROBLEM

In the domain $\Omega=\{(x,t):0<x<1,0<t<T\}$, we consider the following equation

$$ {}_C D_{0t}^{\alpha} u(x,t)+\left[x^{\beta} u_{xx}(x,t)\right]_{xx} = f(x,t), \quad (1)$$

where ${}_C D_{0t}^{\alpha}$ is Caputo fractional differential operator α order [22]

$$ {}_C D_{0t}^{\gamma} g(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{g^{(n)}(z)dz}{(t-z)^{\gamma-n+1}}, \ \left(n=\left[\operatorname{Re}(\gamma)\right]+1, t>0\right),$$

α, β, T are given real numbers, such that $0<\alpha<1,\ 0<\beta<1,\ T>0;\ f(x,t)$ is a given function on Ω.

First, we introduce the definition of the regular solution of the equation (1).

Definition 1. A function $u(x,t)$ satisfying in the domain Ω equation (1) and the following conditions $u(x,t)\in C(\bar{\Omega})$, ${}_C D_{0t}^{\alpha} u(x,t)$, $\left(x^{\beta}u_{xx}\right)_{xx}\in C(\Omega)$ is called regular in the domain Ω solution of the equation (1).

In the domain Ω, we study the following mixed problem for the equation (1):

Problem 1. *Find a regular in the domain Ω solution of the equation (1) satisfying the following initial*

$u(x,0)=\varphi(x)$, $0\le x\le 1$ *(2)*

and the following boundary conditions:

$$u(1,t)=0,\ \frac{\partial u}{\partial x}(0,t)=0,\ \frac{\partial u}{\partial x}(1,t)=0,\ \lim_{x\to 0}\frac{\partial}{\partial x}\left(x^{\beta}\frac{\partial^2 u}{\partial x^2}\right)=0,\ 0<t\le T \quad (3)$$

where $\varphi(x)$ is a given function continuous on $[0,1]$, such that $\varphi(1)=0$, $\varphi'(0)=0, \varphi'(1)=0$, $\lim_{x\to 0}\left[x^{\beta}\varphi''(x)\right]'=0$.

Let us consider homogeneous equation of the equation (1):

$${}_C D_{0t}^{\alpha} u(x,t)=\left[x^{\beta}u_{xx}(x,t)\right]_{xx},\ (x,t)\in\Omega.$$

We will seek a solution of the homogeneous equation satisfying the conditions (3) in the form $u(x,t)=v(x)T(t)$. Then, with respect to the function $T(t)$, we get the following equation

$${}_C D_{0t}^{\alpha} T(t)-\lambda T(t)=0,$$

and with respect to $v(x)$, we get the following spectral problem:

$$Lv=\left[x^{\beta}v''(x)\right]''=\lambda v(x),\ x\in(0,1); \qquad (4)$$

$$v(1)=0,\ v'(0)=0,\ v'(1)=0,\ \lim_{x\to 0}\frac{\partial}{\partial x}\left(x^{\beta}\frac{\partial^2 v}{\partial x^2}\right)=0, \quad (5)$$

i.e. a problem finding those values of the parameter λ for which there exist nontrivial solution of equation (4) satisfying conditions (5).

III. STUDY OF THE SPECTRAL POROBLEM

Now, we study spectral problem {(4),(5)}. Assume that there exists eigenvalues of the spectral problem {(4),(5)}. Under this assumption, first, we will define the sign of the eigenvalue λ. To this end, by multiplying both sides of (4) to the function $v(x)$ and integrating on $[0,1]$, we obtain

$$\lambda\int_0^1 v^2(x)\,dx=\int_0^1\left[x^{\beta}v''(x)\right]''v(x)\,dx.$$

Using the rule of integration by parts two times on the right-hand side of the last equality, we have

$$\lambda\int_0^1 v^2(x)\,dx=\left[\left[x^\beta v''(x)\right]' v(x)-x^\beta v''(x)v'(x)\right]\Big|_0^1+\int_0^1 x^\beta\left[v''(x)\right]^2 dx.$$

Hence, considering (5), we get the following equality

$$\lambda\int_0^1 v^2(x)\,dx=\int_0^1 x^\beta\left[v''(x)\right]^2 dx,$$

which implies that $\lambda\geq 0$.

Let $\lambda=0$, i.e. consider the equation $\left[x^\beta v''(x)\right]''=0$. It is easy to see that the general solution of this equation has the form

$$v(x)=C_1\frac{x^{3-\beta}}{(2-\beta)(3-\beta)}+C_2\frac{x^{2-\beta}}{(1-\beta)(2-\beta)}+C_3x+C_4,$$

where C_j, $j=\overline{1,4}$ are arbitrary constants.

By obeying this function to the condition (5), we get

$$C_1=0,\ C_3=0,\ \frac{C_2}{1-\beta}=0,\ \frac{C_2}{(1-\beta)(2-\beta)}+C_4=0,$$

from which it follows that $C_1=C_2=C_3=C_4=0$. Hence, $v(x)\equiv 0$.

Thus, the problem {(4), (5)} can have nontrivial solution only for positive values of λ.

Now, we will prove the existence of eigenvalues and eigenfunctions of the spectral problem {(4), (5)}. Assuming the right-hand side of (4) is temporarily known function, we construct the Green's function of the problem {(4), (5)}. It is unique and represented as follows

$$G(x,s)=\begin{cases}\dfrac{s^{3-\beta}}{(2-\beta)(3-\beta)}-\dfrac{s^{2-\beta}x^{2-\beta}}{(1-\beta)(2-\beta)^2}+\dfrac{x^{2-\beta}s}{(1-\beta)(2-\beta)}-\\ -\dfrac{s^{2-\beta}}{(2-\beta)^2}-\dfrac{x^{2-\beta}}{(2-\beta)^2}+\dfrac{1}{(2-\beta)^2(3-\beta)}, & x<s,\\ \dfrac{x^{3-\beta}}{(2-\beta)(3-\beta)}-\dfrac{x^{2-\beta}s^{2-\beta}}{(1-\beta)(2-\beta)^2}+\dfrac{s^{2-\beta}x}{(1-\beta)(2-\beta)}-\\ -\dfrac{x^{2-\beta}}{(2-\beta)^2}-\dfrac{s^{2-\beta}}{(2-\beta)^2}+\dfrac{1}{(2-\beta)^2(3-\beta)}, & x>s.\end{cases}$$

Then, using Hilbert's theorem, the problem {(4), (5)} is equivalently reduced to the following integral equation [21]

$$v(x)=\lambda\int_0^1 G(x,s)v(s)\,ds. \qquad (6)$$

Since $G(x,s)$ is symmetric and continuous kernel, then from the theory of integral equations with symmetric kernels [21] it follows that equation (6) [hence problem {(4),(5)} has a countable set of eigenvalues

$$0<\lambda_1<\lambda_2<\lambda_3<...$$

and corresponding orthonormal eigenfunctions

$$v_1(x),v_2(x),v_3(x),...,$$

moreover, any arbitrary function $g(x)\in L_2[0,1]$ expands into a series in these eigenfunctions, which converges on average.

IV. CONVERGENCE OF BASIC BILLINEAR SERIES

Let $\{v_k(x)\}_{k=1}^{\infty}$ and $\{v_k(x)\}_{k=1}^{\infty}$ be the eigenfunctions and eigenvalues of the spectral problem {(4),(5)}, respectively.

In this section, we will prove the convergence of some functional series involving these eigenvalues and eigenfunctions.

Lemma 1. *On the interval* $[0,1]$ *the following series converge uniformly:*

$$\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k},\ \sum_{k=1}^{+\infty}\frac{\left[v_k'(x)\right]^2}{\lambda_k},\ \sum_{k=1}^{+\infty}\frac{\left[x^\beta v_k''(x)\right]^2}{\lambda_k^2},\ \sum_{k=1}^{+\infty}\frac{\left\{[x^\beta v_k''(x)]'\right\}^2}{\lambda_k^2}.$$

Proof. Since, for $0<\beta<1$ the kernel $G(x,s)$ of the integral equation (6) is symmetric, continuous and positive-definite, then based on Mercer's theorem [21], we have

$$G(x,s)=\sum_{k=1}^{+\infty}\frac{v_k(x)v_k(s)}{\lambda_k},$$

particularly

$$\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}=G(x,x)\le const\,. \quad (7)$$

Hence, it follows that the series $\sum_{k=1}^{+\infty}\left[v_k^2(x)/\lambda_k\right]$ converges uniformly.

Now, from the integral equation (6), considering continuity of the function $G(x,s)$, we obtain

$$v_k'(x)=\lambda_k\int_0^1\frac{\partial G(x,s)}{\partial x}v_k(s)ds$$

or

$$v_k'(x)=\lambda_k\int_0^x\frac{\partial G(x,s)}{\partial x}v_k(s)ds+\lambda_k\int_x^1\frac{\partial G(x,s)}{\partial x}v_k(s)ds\,.$$

Using the equation (4), from the last, we have

$$v_k'(x)=\int_0^x \frac{\partial G(x,s)}{\partial x}\left[s^\beta v_k''(s)\right]'' ds+\int_x^1 \frac{\partial G(x,s)}{\partial x}\left[s^\beta v_k''(s)\right]'' ds .$$

Hence, applying the rule of integrating by parts two times, we get

$$v_k'(x)=\left[\frac{\partial G(x,s)}{\partial x}\left[s^\beta v_k''(s)\right]'-\frac{\partial^2 G(x,s)}{\partial x\partial s}s^\beta v_k''(s)\right]_{s=0}^{s=x}+$$

$$+\left[\frac{\partial G(x,s)}{\partial x}\left[s^\beta v_k''(s)\right]'-\frac{\partial^2 G(x,s)}{\partial x\partial s}s^\beta v_k''(s)\right]_{s=x}^{s=1}+\int_0^1 \frac{\partial^3 G(x,s)}{\partial x\partial s^2}s^\beta v_k''(s)ds .$$

Taking $\lim\limits_{s\to 0}\left[s^\beta v_k''(s)\right]'=0$, $\frac{\partial^2 G(x,0)}{\partial x\partial s}=0$, $\frac{\partial^2 G(x,1)}{\partial x\partial s}=0$, $\frac{\partial G(x,1)}{\partial x}=0$ and the continuity of the functions $\frac{\partial G(x,s)}{\partial x}$, $\frac{\partial^2 G(x,s)}{\partial x\partial s}$ for $x=s$, $v_k''(s)$, $\left[s^\beta v_k''(s)\right]'$ into account, we rewrite the last equality in the following form

$$v_k'(x)=\int_0^1 \frac{\partial^3 G(x,s)}{\partial x\partial s^2}s^\beta v_k''(s)ds=\sqrt{\lambda_k}\int_0^1\left[s^{\beta/2}\frac{\partial^3 G(x,s)}{\partial x\partial s^2}\right]\left[\frac{s^{\beta/2}v_k''(s)}{\sqrt{\lambda_k}}\right]ds$$

or

$$\frac{v_k'(x)}{\sqrt{\lambda_k}}=\int_0^1\left[s^{\beta/2}\frac{\partial^3 G(x,s)}{\partial x\partial s^2}\right]\left[\frac{s^{\beta/2}v_k''(s)}{\sqrt{\lambda_k}}\right]ds . \quad (8)$$

It is easy to show that $\left\{s^{\beta/2}v_k''(s)/\sqrt{\lambda_k}\right\}\Big|_{k=1}^{+\infty}$ forms an orthonormal system in $L_2(0,1)$. Indeed,

$$(\lambda_k\lambda_m)^{-1/2}\int_0^1 s^{\beta/2}v_k''(s)\cdot s^{\beta/2}v_m''(s)ds=(\lambda_k\lambda_m)^{-1/2}\int_0^1 s^\beta v_k''(s)v_m''(s)ds=$$

$$=(\lambda_k\lambda_m)^{-1/2}\left\{s^\beta v_k''(s)v_m''(s)-\left[s^\beta v_k''(s)\right]'v_m(s)\right\}\Big|_0^1+(\lambda_k\lambda_m)^{-1/2}\int_0^1\left[s^\beta v_k''(s)\right]''v_m(s)ds=$$

$$=\sqrt{\lambda_k/\lambda_m}\int_0^1 v_k(s)v_m(s)ds=\begin{cases}0, & for\ k\neq m,\\ 1, & for\ \ k=m.\end{cases}.$$

Then, from (8), it follows that the function $v_k'(x)/\sqrt{\lambda_k}$ can be considered as the Fourier coefficient of the function $s^{\beta/2}\frac{\partial^3 G(x,s)}{\partial x\partial s^2}$ with respect to variable s.

If $s^{\beta/2}\frac{\partial^3 G(x,s)}{\partial x\partial s^2}\in L_2[0,1]$ for all x, then based on Bessel's inequality, we obtain

$$\sum_{k=1}^{+\infty}\frac{\left[v_k'(x)\right]^2}{\lambda_k}\le\int_0^1 s^{\beta}\left[\frac{\partial^3 G(x,s)}{\partial x\partial s^2}\right]^2 ds. \quad (9)$$

Using the representation of the function $G(x,s)$, we will show that the right-hand part of the (9) is bounded. Indeed, we have

$$\int_0^1 s^{\beta}\left[\frac{\partial^3 G(x,s)}{\partial x\partial s^2}\right]^2 ds=\int_0^x s^{\beta}\left\{\frac{\partial^3}{\partial x\partial s^2}\left(\frac{x^{3-\beta}}{(2-\beta)(3-\beta)}+\frac{xs^{2-\beta}}{(1-\beta)(2-\beta)}-\frac{s^{2-\beta}x^{2-\beta}}{(2-\beta)^2(1-\beta)}-\right.\right.$$

$$\left.\left.-\frac{s^{2-\beta}}{(2-\beta)^2}-\frac{x^{2-\beta}}{(2-\beta)^2}+\frac{1}{(3-\beta)(2-\beta)^2}\right)\right\}^2 ds+\int_x^1 s^{\beta}\left\{\frac{\partial^3}{\partial x\partial s^2}\left(\frac{s^{3-\beta}}{(2-\beta)(3-\beta)}-\frac{x^{2-\beta}}{(2-\beta)^2}-\right.\right.$$

$$\left.\left.-\frac{s^{2-\beta}x^{2-\beta}}{(2-\beta)^2(1-\beta)}+\frac{x^{2-\beta}s}{(2-\beta)(1-\beta)}\frac{s^{2-\beta}}{(2-\beta)^2}+\frac{1}{(3-\beta)(2-\beta)^2}\right)\right\}^2 ds=$$

$$=\frac{x^{1-\beta}}{1-\beta}-\frac{x^{2-2\beta}}{1-\beta}<\frac{1}{1-\beta}.$$

From the last, we can conclude that series $\sum_{k=1}^{+\infty}\left\{\left[v'(x)\right]^2/\lambda_k\right\}$ converges uniformly.

Since, the function $\frac{\partial G(x,s)}{\partial x}$ is continuous, from the equation (6), we get

$$x^{\beta}v_k''(x)=\lambda_k\int_0^1 x^{\beta}\frac{\partial^2 G(x,s)}{\partial x^2}v_k(s)ds \quad (10)$$

or

$$\frac{x^{\beta}v_k''(x)}{\lambda_k}=\int_0^1 x^{\beta}\frac{\partial^2 G(x,s)}{\partial s^2}v_k(s)ds.$$

Hence, considering $\left[x^{\beta}v_k''(x)/\lambda_k\right]$ as the Fourier coefficient of the function $x^{\beta}\frac{\partial^2 G(x,s)}{\partial x^2}$, based on Bessel's inequality, we obtain

$$\sum_{k=1}^{+\infty}\frac{\left[x^{\beta}v_k''(x)\right]^2}{\lambda_k^2}\le\int_0^1\left[x^{\beta}\frac{\partial^2 G(x,s)}{\partial x^2}\right]^2 ds. \quad (11)$$

We show that the right-hand part of the last equality is limited:

$$\int_0^1\left[x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}\right]^2 ds=\int_0^x\left[x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}\right]^2 ds+\int_x^1\left[x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}\right]^2 ds=$$

$$=\int_0^x x^2-\frac{2x(1-\beta)}{(2-\beta)}+\frac{(1-\beta)^2}{(2-\beta)^2}-\frac{2xs^{2-\beta}}{2-\beta}+\frac{2s^{2-\beta}(1-\beta)}{(2-\beta)^2}+\frac{s^{4-2\beta}}{(2-\beta)^2}ds+$$

$$+\int_x^1\left(s^2-\frac{2s(1-\beta)}{2-\beta}+\frac{(1-\beta)^2}{(2-\beta)^2}-\frac{2s^{3-\beta}}{2-\beta}+\frac{2s^{2-\beta}(1-\beta)}{(2-\beta)^2}+\frac{s^{4-2\beta}}{(2-\beta)^2}\right)ds=$$

$$=\frac{2x^3}{3}-\frac{x^2(1-\beta)}{2-\beta}-\frac{2x^{4-\beta}}{(2-\beta)(3-\beta)(4-\beta)}+\frac{1}{3}-\frac{1-\beta}{2-\beta}+\left(\frac{1-\beta}{2-\beta}\right)^2-$$

$$-\frac{2}{(4-\beta)(2-\beta)}+\frac{2(1-\beta)}{(2-\beta)^2(3-\beta)}+\frac{1}{(2-\beta)^2(5-2\beta)}=$$

$$=\frac{2x^3}{3}+\frac{1}{3}+\left(\frac{1-\beta}{2-\beta}\right)^2+\frac{2(1-\beta)}{(2-\beta)^2(3-\beta)}+\frac{1}{(2-\beta)^2(5-2\beta)}<$$

$$<1+\frac{\left(2\beta^4-15\beta^3+43\beta^2-56\beta+28\right)}{(2-\beta)^2(3-\beta)(5-2\beta)}<1+\frac{2\beta^4+43\beta^3+28}{(2-\beta)^2(3-\beta)(5-2\beta)}.$$

Hence, series $\sum_{k=1}^{+\infty}\left\{\left[x^\beta v_k''(x)\right]^2/\lambda_k^2\right\}$ converges uniformly.

Similarly, from equation (5), taking into account the function $x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}$ is continuous, we get

$$\frac{\left[x^\beta v_k''(x)\right]'}{\lambda_k}=\int_0^1\frac{\partial}{\partial x}\left[x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}\right]v_k(s)ds.$$

Hence, on the basis of Bessel's equality, we have

$$\sum_{k=1}^{+\infty}\frac{\left[\left(x^\beta v_k''(x)\right)'\right]^2}{\lambda_k^2}\le\int_0^1\left\{\frac{\partial}{\partial x}\left[x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}\right]\right\}^2 ds.\ (12)$$

We show that the right hand part of (12) is limited:

$$\int_0^1 \left\{ \frac{\partial}{\partial x}\left[x^\beta \frac{\partial^2 G(x,s)}{\partial x^2} \right] \right\}^2 ds = \int_0^x \left\{ \frac{\partial}{\partial x}\left[\left(x - \frac{s^{2-\beta}}{2-\beta} - \frac{(1-\beta)}{2-\beta} \right) \right] \right\}^2 ds = \int_0^x 1^2 ds = x < 1.$$

Hence it follows that the series $\sum_{k=1}^{+\infty} \left\{ \left[x^\beta v_k''(x) \right]^2 / \lambda_k^2 \right\}$ converges uniformly.

V. THE ORDER OF THE FOURIER COEFFICIENTS

Lemma 1. *Suppose the following conditions hold:*

$$f(x),\ f'(x) \in C[0,1],\ x^{\beta/2} f''(x) \in L_2[0,1],\ f(1) = f'(1) = f'(0) = 0.$$

Then the following Bessel type equality is valid:

$$\sum_{k=1}^{\infty} \lambda_k f_k^2 \le \int_0^1 x^\beta \left[f''(x) \right]^2 dx. \qquad (13)$$

Particularly, it is possible to assert convergence of the series in (13).

Here in after f_k denotes Fourier coefficient of the function $f(x)$ by the system of eigenfunctions $v_k(x)$.

Proof. Let's consider the functional

$$J = \int_0^1 x^\beta \left\{ \left[f(x) - \sum_{k=1}^{n-1} f_k v_k(x) \right]'' \right\}^2 dx = \int_0^1 x^\beta \left[f''(x) \right]^2 dx + \int_0^1 x^\beta \left[\sum_{k=1}^{n-1} f_k v_k''(x) \right]^2 dx -$$

$$-2\int_0^1 x^\beta f''(x) \left[\sum_{k=1}^{n-1} f_k v_k''(x) \right] dx = \int_0^1 x^\beta \left[f''(x) \right]^2 dx + \sum_{k=1}^{n-1} f_k^2 \int_0^1 x^\beta \left[v_k''(x) \right]^2 dx +$$

$$+2\sum_{\substack{k,l=1 \\ k\ne l}}^{n-1} f_k f_l \int_0^1 x^\beta v_k''(x) v_l''(x) dx - 2\sum_{k=1}^{n-1} f_k \int_0^1 x^\beta f''(x) v_k''(x) dx. \qquad (14)$$

Integrating by parts twice, we get

$$\int_0^1 x^\beta v_k''(x) v_l''(x) dx = \left[x^\beta v_k''(x) v_l'(x) - \left[x^\beta v_k''(x) \right]' v_l(x) \right]_{x=0}^{x=1} +$$

$$+\int_0^1 \left[x^\beta v_k''(x) \right]'' v_l(x) dx = \lambda_k \int_0^1 v_k(x) v_l(x) dx = \begin{cases} 0 & for\ \ k \ne l, \\ \lambda_k & for\ \ k = l. \end{cases} \qquad (15)$$

Similarly integrating by parts, we obtain

$$\int_0^1 x^\beta f''(x) v_k''(x) dx = \left[x^\beta v_k''(x) f'(x) - \left[x^\beta v_k''(x) \right]' f(x) \right]_{x=0}^{x=1} +$$

$$+\int_0^1 \left[x^\beta v_k''(x)\right]'' f(x)dx = \lambda_k \int_0^1 f(x)v_k(x)dx = \lambda_k f_k. \quad (16)$$

By virtue of (15) and (16), from (14) it follows that

$$J = \int_0^1 x^\beta \left[f''(x)\right]^2 dx - \sum_{k=1}^{n-1} \lambda_k f_k^2 \geq 0 \text{ for } \forall n \in N.$$

From the last inequality, it follows that the fairness of the inequality (14).

Lemma 2. Suppose that the following conditions hold:

$$f(x),\ f'(x),\ x^\beta f''(x),\ \left[x^\beta f''(x)\right]' \in C[0,1],\ \left[x^\beta f''(x)\right]'' \in L_2[0,1],$$

$$f(1) = f'(0) = f'(1) = 0,\ \lim_{x\to 0} \frac{\partial}{\partial x}\left(x^\beta \frac{\partial^2 f}{\partial x^2}\right) = 0.$$

Then the following Bessel type equality is valid:

$$\sum_{k=1}^{\infty} \lambda_k^2 f_k^2 \leq \int_0^1 \left[\left(x^\beta f''(x)\right)''\right]^2 dx. \quad (17)$$

Particularly, it is possible to assert convergence of the series in (17).

Proof. By virtue of the conditions of lemma 2 equality (16) is fulfilled. In addition, the following equality is valid as

$$\int_0^1 x^\beta f''(x)v_k''(x)dx = \int_0^1 \left[x^\beta f''(x)\right]'' v_k(x)dx. \quad (18)$$

Indeed, integrating by parts the left hand part of the last equality gives

$$\int_0^1 x^\beta v_k''(x) f''(x)dx = \left\{x^\beta f''(x)v_k'(x) - \left[x^\beta f''(x)\right]' v_k(x)\right\}\Big|_{x=0}^{x=1} +$$

$$+\int_0^1 \left[x^\beta f''(x)\right]'' v_k(x)dx.$$

Taking into account $x^\beta f''(x)$, $\left[x^\beta f''(x)\right]'$ are continuous on $[0,1]$, $\left[x^\beta f''(x)\right]'\Big|_{x=0} = 0,\ v_k'(0) = 0$, and also conditions $f''(1) = f'''(1) = 0$, we get

$$\left[x^\beta f''(x)v_k''(x) - \left[x^\beta f_k''(x)\right]' v_k(x)\right]_{x=0}^{x=1} = 0.$$

From (16) and (18) it follows that

$$\int_0^1 \left[x^\beta f''(x)\right]'' v_k(x)dx = \lambda_k f_k .$$

Hence, $\lambda_k f_k$ is the Fourier coefficient of the function $Lf \equiv \left[x^\beta f''(x)\right]''$, or

$$(Lf)_k = \left[\left(x^\beta f''(x)\right)''\right]_k = \lambda_k f_k . \quad (19)$$

Writing Bessel's inequality for function $\left[x^\beta f''(x)\right]''$ and taking equality (19) into account, we get sought inequality.

VI. JUSTIFICATION OF THE METHOD OF FOURIER

Theorem 1. Suppose that the following conditions hold:

1. $\varphi(x)$ satisfies conditions of the lemma 1.

2. $f(x,t)$ satisfies conditions of the lemma 1 uniformly with respect to t

3. $f_t'(x,t)$ satisfies conditions of the lemmas 1 and 2 uniformly with respect to variable t.

Then the following

$$u(x,t) = \sum_{k=1}^{+\infty}\left[\varphi_k E_{\alpha,1}\left(-\lambda_k t^\alpha\right) + \int_0^t (t-z)^{\alpha-1} E_{\alpha,\alpha}\left[-\lambda_k (t-z)^\alpha\right] f_k(z)dz\right] v_k(x) \quad (20)$$

series will be regular solution of the problem $\{(1),(2),(3)\}$*,where* $E_{\alpha,\beta}(z)$ *is two-parameter Mittage-Leffler's function[]:*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} .$$

Proof. We shall seek solution of the considered in the form

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) v_k(x) \quad (21)$$

where $v_k(x)$, $k \in N$ are functions which are defined by (7), and $T_k(t)$, $k \in N$ are unknown functions.

Expanding functions $f(x,t)$ and $\varphi(x)$ to series by functions $v_k(x)$, we get

$$f(x,t) = \sum_{k=1}^{\infty} f_k(t) v_k(x),\ \varphi(x) = \sum_{k=1}^{\infty} \varphi_k v_k(x) \quad (22)$$

where

$$f_k(t) = \int_0^1 f(x,t) v_k(x)dx,\ \varphi_k = \int_0^1 \varphi(x) v_k(x)dx . \quad (23)$$

Substituting (21) and (22) into equation (1) and the condition (2), we obtain

$$\sum_{k=1}^{+\infty} {}_C D_{0t}^{\alpha} T_k(t)\cdot v_k(x) + \sum_{k=1}^{+\infty} T_k(t)\left[x^{\beta} v_k''(x)\right]'' = \sum_{n=1}^{\infty} f_k(t) v_k(x),$$

$$\sum_{k=1}^{+\infty} T_k(0)\cdot v_k(x) = \sum_{k=1}^{\infty} \varphi_k \cdot v_k(x).$$

From the last equalities, by virtue of (4) and completeness of the system of functions $v_k(x)$, $k\in N$ in the space $L_2(0,1)$, it follows the following problem with respect to unknown function $T_k(t)$:

$${}_C D_{0t}^{\alpha} T_k(t) + \lambda_k T_k(t) = f_k(t),\; T_k(0) = \varphi_k \;.$$

It is known the solution of this problem represented as follows [22]

$$T_k(t) = E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k + \int_0^t (t-z)^{\alpha-1} E_{\alpha,\alpha}\left[-\lambda_k (t-z)^{\alpha}\right] f_k(z)\,dz\,.$$

Substituting this expression of $T_k(t)$ into (21), we obtain formula (20).

Now, we show uniformly convergence of the series

$$u(x,t),\; u_x(x,t),\; x^{\beta} u_{xx}(x,t),\; \left[x^{\beta} u_{xx}(x,t)\right]_x,\; {}_C D_{0t}^{\alpha} u(x,t).$$

Using the rule of integrating by parts we rewrite (20) as follows

$$u(x,t) = \sum_{k=1}^{+\infty} E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k v_k(x) + \sum_{k=1}^{+\infty} E_{\alpha,\alpha+1}\left(-\lambda_k t^{\alpha}\right) f_k(0) v_k(x) +$$

$$+\sum_{k=1}^{+\infty} \int_0^t (t-z)^{\alpha} E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^{\alpha}\right] f_k'(z)\,dz \cdot v_k(x). \quad (24)$$

We show absolute and uniformly convergence of the series in the right hand side of (24).

Since the Mittag-Leffler function is bounded [22], i.e.

$$\left|E_{\alpha,\beta}(-z)\right| \le M\,,\; 0 < M < +\infty,\; z > 0\,, \quad (25)$$

one can easily verify that the following inequalities are valid:

$$\sum_{k=1}^{+\infty} \left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k \cdot v_k(x)\right| \le M\left[\sum_{k=1}^{+\infty} \lambda_k \varphi_k^2 \cdot \sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty} f_k(0) t^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda_k t^{\alpha}\right) v_k(x) \le M\cdot T^{\alpha}\left[\sum_{k=1}^{+\infty} \lambda_k f_k^2(0)\cdot \sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\int_0^t (t-z)^{\alpha} E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^{\alpha}\right] f_k'(z)dz\cdot v_k(x)\le$$

$$\le M\cdot\sqrt{\frac{T^{2\alpha+1}}{2\alpha+1}}\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k f_k'^2(z)dz\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2}.$$

We show the validity of the first inequality. Using (25) and Cauchy-Schwarz inequality, we get

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k\cdot v_k(x)\right|\le M\sum_{k=1}^{+\infty}\left|\varphi_k\cdot v_k(x)\right|=$$

$$=M\sum_{k=1}^{+\infty}\left|\sqrt{\lambda_k}\varphi_k\cdot\frac{v_k(x)}{\sqrt{\lambda_k}}\right|\le M\left[\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2}.$$

Similarly, one can show validity of the second and third inequalities.

From these inequalities, taking into account lemma 1 and lemma 2, it follows that absolutely and uniformly convergence of the series of (24) and $u(x,t)\in C(\bar{\Omega})$.

Differentiating (25) with respect to x, we obtain

$$u_x(x,t)=\sum_{k=1}^{+\infty}E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k v_k'(x)+\sum_{k=1}^{+\infty}E_{\alpha,\alpha+1}\left(-\lambda_k t^{\alpha}\right) f_k(0)v_k'(x)+$$

$$+\sum_{k=1}^{+\infty}\int_0^t (t-z)^{\alpha} E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^{\alpha}\right] f_k'(z)dz\cdot v_k'(x).\ (26)$$

Analogously, it can be shown that the following inequalities hold:

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k v_k'(x)\right|\le M\left[\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2\cdot\sum_{k=1}^{+\infty}\frac{v_k'^2}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right) f_k(0)v_k'(x)\right|\le M\cdot T^{\alpha}\left[\sum_{k=1}^{+\infty}\lambda_k f_k^2(0)\cdot\sum_{k=1}^{+\infty}\frac{v_k'^2}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\int_0^t (t-z)^{\alpha} E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^{\alpha}\right] f_k'(z)dz\cdot v_k'(x)\le$$

$$\le M\cdot\sqrt{\frac{T^{2\alpha+1}}{2\alpha+1}}\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k f_k'^2(z)dz\cdot\sum_{k=1}^{+\infty}\frac{v_k'^2(x)}{\lambda_k}\right]^{1/2}.$$

From the last inequalities by virtue of the conditions of Theorem 1, it follows that the absolute and uniformly convergence of the series in (26).

Now, differentiating (26) with respect to x and multiplying the both sides of the taken equality to x^{β}, we obtain

$$x^{\beta}u_{xx}(x,t)=\sum_{k=1}^{+\infty}E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k x^{\beta}v_k''(x)+\sum_{k=1}^{+\infty}E_{\alpha,\alpha+1}\left(-\lambda_k t^{\alpha}\right)f_k(0)x^{\beta}v_k''(x)+$$

$$+\sum_{k=1}^{+\infty}\int_0^t (t-z)^{\alpha}E_{\alpha,\alpha+1}\left[-\lambda_k(t-z)^{\alpha}\right]f_k'(z)dz\cdot x^{\beta}v_k''(x). \quad (27)$$

We show the convergence of the series in (27). Firstly, we show that the following inequalities are valid:

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k x^{\beta}v_k''(x)\right|\le M\left[\sum_{k=1}^{+\infty}\lambda_k^2\varphi_k^2\cdot\sum_{k=1}^{+\infty}\frac{\left[x^{\beta}v_k''(x)\right]^2}{\lambda_k^2}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)t^{\alpha}f_k(0)x^{\beta}v_k''(x)\right|\le M\cdot T^{\alpha}\left[\sum_{k=1}^{+\infty}\lambda_k^2 f_k^2(0)\cdot\sum_{k=1}^{+\infty}\frac{\left[x^{\beta}v_k''(x)\right]^2}{\lambda_k^2}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\int_0^t (t-z)^{\alpha}E_{\alpha,\alpha+1}\left[-\lambda_k(t-z)^{\alpha}\right]f_k'(z)dz\cdot x^{\beta}v_k''(x)\le$$

$$\le M\cdot\sqrt{\frac{T^{2\alpha+1}}{2\alpha+1}}\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k^2 f_k'^2(z)dz\cdot\sum_{k=1}^{+\infty}\frac{x^{2\beta}v_k''^2(x)}{\lambda_k^2}\right]^{1/2}.$$

Let’s prove the first inequality. Taking into account (25), we get

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k x^{\beta}v_k''(x)\right|\le M\sum_{k=1}^{+\infty}\left|\varphi_k x^{\beta}v_k''(x)\right|=M\sum_{k=1}^{+\infty}\left|\lambda_k\varphi_k\cdot\frac{x^{\beta}v_k''(x)}{\lambda_k}\right|.$$

Applying Cauchy-Schwarz inequality from the last, we obtain proof of the first inequality. The second and third inequalities are proved similarly.

Using the same method one can prove absolute and uniformly convergence of the series in $\left[x^{\beta}u_{xx}(x,t)\right]_x$.

Now, we investigate ${}_C D_{0t}^{\alpha}u(x,t)$. Applying differential operator ${}_C D_{0t}^{\alpha}$ to the both sides of (24) and taking into account ${}_C D_{0t}^{\alpha}T_k(t)=-\lambda_k T_k(t)$ and also the view of the function $T_k(t)$, we find

$${}_C D_{0t}^{\alpha}u(x,t)=-\sum_{k=1}^{+\infty}\lambda_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k v_k(x)-\sum_{k=1}^{+\infty}\lambda_k E_{\alpha,\alpha+1}\left(-\lambda_k t^{\alpha}\right)f_k(0)v_k(x)-$$

$$-\sum_{k=1}^{+\infty}\lambda_k\int_0^t (t-z)^{\alpha}E_{\alpha,\alpha+1}\left[-\lambda_k(t-z)^{\alpha}\right]f_k'(z)dz\cdot v_k(x).$$

And convergence of the series in ${}_C D_{0t}^{\alpha}u(x,t)$ follows by the following inequalities

$$\sum_{k=1}^{+\infty}\left|\lambda_k E_{\alpha,1}\left(-\lambda_k t^\alpha\right)\varphi_k v_k(x)\right| \le M\left[\sum_{k=1}^{+\infty}\lambda_k^3\varphi_k^2\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|\lambda_k E_{\alpha,1}\left(-\lambda_k t^\alpha\right)t^\alpha f_k(0)v_k(x)\right| \le MT^\alpha\left[\sum_{k=1}^{+\infty}\lambda_k^3 f_k^2(0)\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|\lambda_k\int_0^t (t-z)^\alpha E_{\alpha,\alpha+1}\left[-\lambda_k(t-z)^\alpha\right]f_k'(z)dz\cdot v_k(x)\right| \le$$

$$\le M\cdot\sqrt{\frac{T^{2\alpha+1}}{2\alpha+1}}\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k^3 f_k'^2(z)dz\cdot\sum_{k=1}^{+\infty}\frac{v^2(x)}{\lambda_k}\right]^{1/2}.$$

These inequalities are proved the same method as above-proved inequalities. From these inequalities, it follows the absolute and uniform convergence of the series in ${}_C D_{0t}^\alpha u(x,t)$.

Theorem 1 has been proved.

VII. THE UNIQUENESS OF THE SOLUTION OF THE PROBLEM

Now, we prove the uniqueness of the solution of the problem. For this aim, we introduce the following function

$$T_k(t)=\int_0^1 u(x,t)v_k(x)dx. \quad (28)$$

Based on (27), we consider the following auxiliary function

$$T_{k,\varepsilon}(t)=\int_\varepsilon^{1-\varepsilon} u(x,t)v_k(x)dx, \quad (29)$$

where ε is sufficiently small positive number.

Applying differential operator ${}_C D_{0t}^\alpha$ to (29) and using homogeneous equation corresponding equation (1), we get

$${}_C D_{0t}^\alpha T_{k,\varepsilon}(t)=-\int_\varepsilon^{1-\varepsilon}\left[x^\beta u_{xx}(x,t)\right]_{xx}\cdot v_k(x)dx.$$

Using the rule of integration by parts four times from the last equality, we get

$${}_C D_{0t}^\alpha T_{k,\varepsilon}(t)=-\Big\{v_k(x)\left[x^\beta u_{xx}(x,t)\right]_x - v_k'(x)x^\beta u_{xx}(x,t)+x^\beta v_k''(x)u_x(x,t)-$$

$$-\left[x^\beta v_k''(x)\right]' u(x,t)\Big\}\Big|_{x=\varepsilon}^{x=1-\varepsilon}-\int_\varepsilon^{1-\varepsilon}\left[x^\beta v_k''(x)\right]'' u(x,t)dx. \quad (30)$$

Passing to the limit as $\varepsilon\to 0$ and taking (4) and (28) into account, from (30), we derive

$${}_C D_{0t}^\alpha T_k(t)+\lambda_k T_k(t)=0. \quad (31)$$

From (28), we have

$$T_k(0)=\int_0^1 u(x,0)v_k(x)dx=\varphi_k . \quad (32)$$

It is known that, the solution of the equation (31) satisfying condition (32) is represented as follows

$$T_k(t)=\varphi_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right).$$

Let $\varphi(x)\equiv 0$, $x\in[0,l]$. Then from the last equality taking into account (28) for all $t\in[0,\mathrm{T}]$ and $k\in N$ it follows that

$$\int_0^1 u(x,t)v_k(x)dx=0 . \quad (33)$$

As since problem {(4),(5)} is self-adjoint, its eigenfunctions will be complete system in $L_2[0,l]$. Taking this account from (33), we get $u(x,t)\equiv 0$ almost everywhere on $[0,l]$ for all $t\in[0,T]$. By virtue of $u(x,t)\in C(\bar{\Omega})$, we obtain $u(x,t)\equiv 0$ in $\bar{\Omega}$. Thus, homogeneous problem has only trivial solution and this gives us the uniqueness of the solution of the considered problem.

CONCLUSION

In the present paper, we consider an initial-boundary value problem for a fourth-order, time-fractional, space-degenerate partial differential equation in a rectangular domain. By applying the method of separation of variables, we obtain a spectral problem for an ordinary differential equation. Since this equation has a degenerate coefficient, we cannot find the eigenvalues and eigenfunctions of this spectral problem in explicit form. However, by applying the theory of integral equations with symmetric kernels, we establish the existence and some properties of the eigenvalues and eigenfunctions of this spectral problem. Using these properties, we prove the existence and uniqueness of the main problem.

REFERENCES

1. Dzhrbashyan M.M., Nersesyan A.B. Fractional derivatives and the Cauchy problem for differential equations of fractional order // Izv. AN ArmSSR. Mat. 1968. Vol. 3. No. 1. pp 3-29 (in Russian).
2. Dzhrbashyan M.M. Boundary value problem for a differential operator of fractional order of the Sturm-Liouville type // Izv. AN ArmSSR. Mat. 1970. Vol. 5. No. 2. pp 71-96 (in Russian).
3. Nakhushev A.M. The Sturm-Liouville problem for a second-ordr ordinary differential equation with fractional derivatives in lower terms, Report AN SSSR, 234:2(1977), 308-311 (in Russian).
4. Aleroev T. S. On the problem of the zeros of the Mittag-Leffler function and the spectrum of a differential operator of fractional order, Differ. equations, (2000) 36. no.9, 1278–1279 (in Russian).
5. Pskhu A.V. Equations in partial derivatives of fractional order. Moscow. Nauka, 2005 (in Russian).
6. Nakhushev A.M. Fractional calculus and its application. Moscow. Fizmatlit, 2003 (in Russian).

7. Samko S. G., Kilbas A. A., Marichev O. I. Integrals and derivatives of the fractional order and some of their applications. Minsk. Science and technology, 1987. (in Russian).
8. Berdyshev A.S., Cabada A., Kadirkulov B.J.. The Samarskii–Ionkin type problem for the fourth order parabolic equation with fractional differential operator. Computers and Mathematics with Applications. (2011). no 62. 3884-3893.
9. Berdyshev A. S., Kadirkulov B. J. A Samarskii-Ionkin problem for two-dimensionalparabolic equation with the caputo fractional differential operator. International Journal of Pure and Applied Mathematics. 113, (2017), no.4, 53-64.
10. Kerbal S., Kadirkulov B. J. , Kirane M. Direct and inverse problems for a Samarskii-Ionkin type problem for a two dimensional fractional parabolic equation. 2018. No. 3. pp. 147-160.
11. Aziz S., Malik S. A. Identifcation of an unknown source term for a time fractional fourth-order parabolic equation. Electron. J. Differ. Equat. 293. (2016), 1–20.
12. Berdyshev A. S., Kadirkulov B. J. On a nonlocal problem for a fourth-order parabolic equation with a fractional by the Dzhrbashyan–Nersesyan operator. Differential equations. 52 (2016), no. 1, 123–127 (in Russian).
13. L.L. Karashev. Half-strip problem for a parabolic equation fourth order with the Riemann–Liouville operator by temporary variable. Proceedings of the Kabardino-Balkarian Scientific Center of the Russian Academy of Sciences. 91(2019). no.5, 21-29 (in Russian).
14. Yuldashev T. K., Kadirkulov B. J. Nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator. Ural mathematical journal. 6. (2020). no. 1. 153–167. DOI: 10.15826/umj.2020.1.013
15. Yuldashev T. K., Kadirkulov B.J. Inverse boundary value problem for a fractional differential equations of mixed type with integral redenition conditions . Lobachevskii Journal of Mathematics, 42 (2021), no. 3, 649–662.
16. Ashurov R., Umarov S. Determination of the order of fractional derivative for subdiffusion equations. Fract. Calc. Appl. Anal. 23.(2020), no 6, 1647–1662. DOI: 10.1515/fca-2020-0081.
17. Ashurov R., Fayziev Y. Inverse Problem for Finding the Order of the Fractional Derivative in the Wave Equation. Mathematical Notes, 110(2021), no.6, 842–852.
18. Karimov D. X., Kasimova M. A mixed problem for a fourth-order linear equation that degenerates at the boundary of a domain. Izv. Academy of Sciences of the Uzbek SSR, ser.phys.-math.sci. (1968),no. 2, 27-31 (in Russian).
19. Baikuziev K.B., Kasimova M. Mixed problem of the fourth order equation, degenerating at the boundary of the region. Izv. Academy of Sciences of the Uzbek SSR, ser. physical -mat. Nauk, (1968), no.5, 7-12(in Russian).
20. Kasimova M. A mixed problem for a fourth-order linear equation, degenerating at the boundary of the region. Izv. Academy of Sciences of the Uzbek SSR, ser. physical -mat. Nauk, (1968), no. 5, 35-39(in Russian).
21. Mikhlin S. G. Linear Integral Equations. Moscow: Fizmatgiz, 1959.
22. I.Podlubny. Fractional differential equations. Academic Press. 1999.