E-ISSN: 2997-9439

SemantJournals

American Journal of Education and Evaluation Studies
https://semantjournals.org/index.php/ AJEES

| Research Article

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Classification of Simplex Sections Defined by a Hyperplane

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Abstract: In a number of applied problems [6]-[8] the points of the simplex are considered as states of some biological (physical, economic, etc.) system. The transition from one state to another is specified by an evolutionary operator, which can be a differential equation (with or without memory) or a difference equation. Depending on the parameters, the evolution of the system can occur only on some hyperplane intersecting the simplex [1]. In this case, the problem of determining the type of the resulting polyhedron arises.

Keywords: Simplex, section of simplex, convex independence.

1. Introduction

Let R^m given n point $x_1, x_2, ..., x_n$ and

$$co\{x_1, x_2, ..., x_n\} = \left\{x : \ x = \sum_{i=1}^{n} \lambda_i x_i, \lambda_i \geqslant 0, \ \sum_{i=1}^{n} \lambda_i = 1\right\}$$

the convex hull of these points. If f is a non-zero linear functional, then the hyperplane

$H_0 = \{x : f(x) = d\}$ separates the points x and y under the condition $(f(x) - d)(f(y) - d) < 0$.

Definition 1. Points $x_1, x_2, ..., x_n$ are convexly independent if none of them belongs to the convex hull of the others.

Thus, if $x_1, x_2, ..., x_n$ are convexly independent, then any of them can be separated from the others. Obviously, the converse is also true, i.e. if any point can be separated from the others, then these points are convexly independent.

2. Main Part

Let F be a convex polyhedron in R^m, $H_0 = \{x : f(x) = d\}$, $H_+ = \{x : f(x) > d\}$ $H_- = \{x : f(x) < d\}$. Then $F \cap H_0$ is called the section of F by the hyperplane H_0. Let $F_+ = F \cap H_+$ и $F_- = F \cap H_-$.

Theorem 1. If F is a convex polyhedron, then F+ and F− are connected (or empty) sets.

Since the intersection of convex sets is convex, the proof follows directly from the definitions.

Corollary 1. Let $x_1, x_2, ..., x_n$ *vertices of a convex polyhedron F. Then any vertex can be separated from the other vertices by some hyperplane.*

Indeed, the vertices of a convex polyhedron are convexly independent.

Remark 1. It is clear that two vertices of a square that are the ends of a diagonal cannot be separated from the other two vertices.

Definition 2. Points $x_1, x_2, ..., x_n$ *are in general position if the vectors* $x_2 - x_1, x_3 - x_1, ..., x_n - x_1$ *linearly independent.*

Convex hull of points $x_1, x_2, ..., x_n$ located in general position is called an n 1-dimensional simplex, and the points $x_1, x_2, ..., x_n$ vertices of the simplex. From the definition it is clear that the vertices of the simplex are convexly independent.

Let $x_1, x_2, ..., x_n$ vertices of the simplex. Then the convex hull of any k vertices is called $k - 1-$ dimensional face.

Theorem 2. If any two vertices of a convex polyhedron F can be separated from the other vertices, then all vertices of F are in general position.

Proof. If some two-dimensional face F contains more than three vertices, then there exists a pair of vertices not connected by an edge. In this case, they cannot be separated from the other vertices, since they themselves form a disconnected set. It is clear that a two-dimensional face F cannot contain less than three vertices, and these three vertices are in general position. Without loss of generality, assume that a convex polyhedron $F \subset R^m$ has a dimension m и $x_1, x_2, ..., x_n$ its peaks. Clearly $n \geq m + 1$. Let us assume that n > m + 1. Then the vectors $x_2\ x_1, x_3\ x_1, ..., x_n\ x_1$ linearly dependent, since their number exceeds m.

Since dimF = m, then from the vertices $x_1, x_2, ..., x_n$ you can choose m + 1 piece, let's say, $x_1, x_2, ..., x_{m+1}$ so that they are in a common position. Let $F_m = co\ x_1, x_2, ..., x_{m+1}$. So, $n > m + 1$, then x_{m+2} / F_m.

Let's consider straight lines $x_1x_{m+2}, x_2x_{m+2}, ..., x_{m+1}x_{m+2}$. According to M. Pasha's axiom, at least one of these lines contains a point belonging to Fm and different from the points $x_1, x_2, ..., x_{m+1}$.

If such a straight line is x_ix_{m+2}, then the segment connecting the points x_i andx_{m+2} cannot be an edge for F , since the intersection of any two edges of a convex polyhedron is either empty or one of the vertices. Therefore, the vertices x_i и x_{m+2} cannot be separated from other vertices F .

Corollary 2. Among all convex polyhedra, only the simplex can have any two vertices separated from the other vertices.

Corollary 3. In a simplex, any number of vertices can be separated from the rest.

Proof. Let $x_1, x_2, ..., x_{m+1}$ vertices of the simplex and $1 < k < m + 1$.

Let's consider the faces of the simplex $K_1 = co\{x_1, x_2, ..., x_k\}$ and $K_2 = co\{x_{k+1}, x_{k+2}, ..., x_{m+1}\}$

It is easy to see that K_1 and K_2 non-intersecting convex compact sets. Therefore, they can be separated by some hyperplane.

Definition 3. Two convex polyhedra F_1 *and* F_2 *dimensions* m *in space* R^m *affinely homeomorphic if there exists a non-singular matrix* A *and a vector* $y \in R^m$ *such that the mapping* $Ax + y$ *translates* F_1 *to* F_2

It is known [2]-[4] that under an affine transformation, parallel lines become parallel and intersecting lines become intersecting, and also

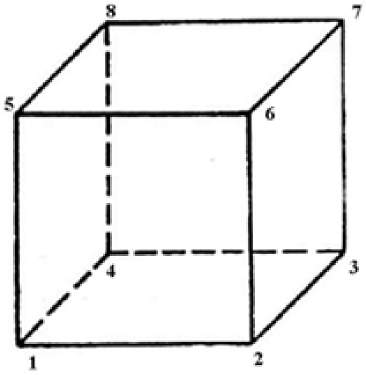

Fig. 1

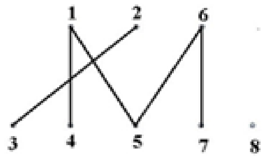

Fig. 2.

on plane R^2 any two triangles are affine homeomorphic. The proof of the last statement can easily be extended to the case of arbitrary dimension, namely, any two simplexes of the same dimension are affine homeomorphic, since the vertices of the simplex are in general position.

Theorem 3. Any hyperplane separating one of the vertices of the simplex from the others in a section with the simplex forms a simplex whose dimension is 1 less than the dimension of the original simplex.

Proof. Let x_1, x_2, ..., x_{m+1} vertices of the simplex and a hyperplane separates the vertex x1 from the others. Then the vertices x_1 and x_k где $k = 2, 3, ..., m + 1$ are located in different open half-spaces defined by the hyperplane. Therefore, there are numbers $0 < t_k < 1$ such that $t_k x_1 + (1 - t_k) x_k$ belong to the hyperplane for all $k = 2, 3, ..., m + 1$.

Since vectors x_2 x_1, ..., x_{m+1} x_1 are linearly independent, then the vectors y_2 x_1, y_3 x_1, ..., y_{m+1} x_1, where $y_k = t_k x_1 + (1-t_k) x_k$ ($k = 2, 3, ..., m + 1$) are also linearly independent, otherwise the original system x_2 x_1, ..., x_{m+1} x_1 would be linearly dependent. Therefore, $x_1, y_2, ..., y_{m+1}$ are the vertices of a simplex, and the section is the face of this simplex stretched across the vertices $y_2, ..., y_{m+1}$. Thus, the section is a simplex of dimension m − 1.

Corollary 4. Any two sections of an m-dimensional simplex by hyperplanes separating one of the vertices of an affine are homeomorphic.

Example. Let x_1, x_2, x_3, x_4 vertices of a three-dimensional simplex. It is clear that any plane separating two vertices from two others in a section defines a quadrilateral. However, two quadrilaterals on a plane are not necessarily affinely homeomorphic, for example, a square and a trapezoid are not.

Definition 4. A graph is called bipartite if the set of its vertices can be divided into two non-empty and disjoint classes so that the vertices of any arc of this graph belong to different classes.

Example. Let the vertices of a cube be marked with numbers 1, 2, ..., 8 and the plane separates vertices 1, 2, 6 from the remaining vertices.

In this case, we obtain the following bipartite graph Fig. 2.

In Fig. 2. only those edges of the cube whose ends lie in different half-spaces defined by the cutting plane are preserved. It is clear that the bipartite graph corresponding to the section of the cube cannot contain more than six arcs, since the section of the cube is a polygon with no more than six sides. It is also obvious that the vertices of the cube 1 and 3 (Fig. 1.) cannot be separated from other vertices.

Thus, the section of a convex m-dimensional polyhedron by a hyperplane that does not contain the vertices of the polyhedron determines a certain bipartite graph. Note that the converse, generally speaking, is not true.

Definition 5. A bipartite graph is called complete if any two vertices belonging to different classes are connected by an arc.

Theorem 4. An arbitrary complete bipartite graph uniquely determines a section of a simplex by a hyperplane that does not contain a vertex of this simplex.

Proof. Let 1, 2, 3, ..., m + 1 be the vertices of a bipartite graph G, where vertices 1, 2, 3, ..., k belong to class I, and the remaining vertices belong to class , where . We also denote the vertices of the dimensional simplex by the numbers 1, 2, 3, ..., m + 1. It is clear that the face of the simplex with vertices 1 and k + 1, k + 2, ..., m + 1 is also a simplex of dimension m k + 1, where vertex 1 is separated from vertices of class II. According to Corollary 2 and Theorem 3, there exists a hyperplane H0, which in intersection with this face defines a simplex of dimension m k. Let this intersection be 1. Similarly, we define 2, 3, ..., k . Obviously, these simplices do not contain vertices 1, 2, 3, ..., m + 1. Therefore, any two simplices 1, 2, ..., k do not intersect, since their intersection could belong only to the face containing all vertices from class II. Then at least one vertex of the intersection would belong to the convex hull of class II.

Corollary 5. The section of the simplex by the hyperplane separating k vertices from the rest is the convex hull of k disjoint simplices of dimension m − k.

Definition 6. Let F be a convex m− dimensional polyhedron. The set of numbers $(t_0, t_1, ..., t_{m-1})$, *where* t_i *the number i of dimensional faces of F will be called the type of this polyhedron* Example: Let x_1, x_2, x_3, x_4, x_5 vertices of a four-dimensional simplex. If the hyperplane defines two of them from the remaining vertices, then we obtain the following bipartite graph

Since the vertices of the edge x_1x_3 belong to different classes, then the edge x_1x_3 contains one of the vertices of the section. Since there are 6 such edges, the section has 6 vertices. It is clear that the face $x_1x_2x_3$ defines the edge of the section. Obviously, there are 9 such two-dimensional faces. Therefore, the section has 9 edges. Further, the faces $x_1x_3x_4x_5$ and $x_2x_3x_4x_5$ defines one two-dimensional face of the section. In these cases, two-dimensional

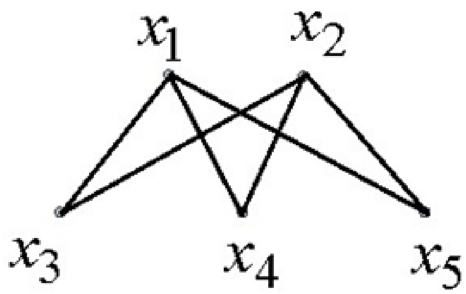

Fig. 3.

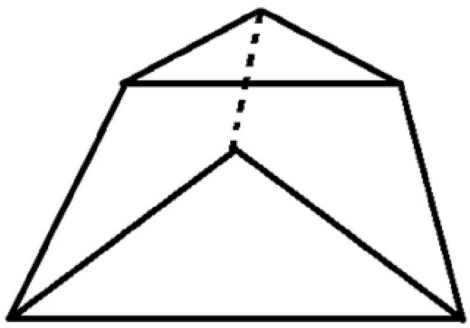

Fig. 4.

the faces of the section are triangles. The faces $x_1x_2x_3x_4$, $x_1x_2x_3x_5$, $x_1x_2x_4x_5$ also defines sections by one two-dimensional face, which are quadrangles. Thus, the section has 5 two-dimensional faces, of which two are triangles and three are quadrangles.

Under the conditions of the example, the section has the following form Fig. 4.

Thus, the section has the type (6; 9; 5), and each quadrangle is located on the same plane. Simple combinatorial calculations [5] allow us to move on to the general case.

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